



Attractors for a deconvolution model of turbulence

Roger Lewandowski, Yves Preaux

► To cite this version:

Roger Lewandowski, Yves Preaux. Attractors for a deconvolution model of turbulence. Applied Mathematics Letters, 2009, 22 (5), pp.642-645. 10.1016/j.aml.2008.05.007 . hal-00271884

HAL Id: hal-00271884

<https://hal.science/hal-00271884>

Submitted on 10 Apr 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Attractors for a deconvolution model of turbulence

Roger Lewandowski* and Yves Preaux†

April 10, 2008

Abstract

We consider a deconvolution model for 3D periodic flows. We show the existence of a global attractor for the model.

MCS Classification : 76D05, 35Q30, 76F65, 76D03

Key-words : Navier-Stokes equations, Large eddy simulation, Deconvolution models.

1 Introduction

This note is concerned by the deconvolution model of order N introduced in [6] (model (2.7) below) for 3D periodic flows. This model takes inspiration in the class of the so called α -models (see in [2] and [4] and references inside) and also in the class of ADM models (see in [7]). We are interested by the question of the existence of a global attractor for this model.

The question of attractors has already been considered for the alpha model (see [1]), corresponding to the case $N = 0$. We prove in this work the existence of an attractor for each N (see Theorem 3.1).

In order to make the paper self contained, we describe carefully how is constructed the deconvolution model. Next, we recall basic notions on the attractors, notions that can be founded in the book of R. Temam (see [8]). Finally we prove the existence of the attractor. The question of its dimension is under progress.

2 The Deconvolution model

2.1 Function Spaces

for $s \in \mathbb{R}$, let us define the space function

$$(2.1) \quad \mathbf{H}_s = \left\{ \mathbf{w} = \sum_{\mathbf{k}} \widehat{\mathbf{w}} e^{i\mathbf{k} \cdot \mathbf{x}}, \nabla \cdot \mathbf{w} = 0, \widehat{\mathbf{w}}(\mathbf{0}) = \mathbf{0}, \sum_{\mathbf{k}} |\mathbf{k}|^{2s} |\widehat{\mathbf{w}}(\mathbf{k}, t)|^2 < \infty \right\}.$$

*IRMAR, UMR 6625, Université Rennes 1, Campus Beaulieu, 35042 Rennes cedex FRANCE; Roger.Lewandowski@univ-rennes1.fr, <http://perso.univ-rennes1.fr/roger.lewandowski/>

†Lycée du Puy de Lôme, rue du Puy de Lôme, 29 200 Brest, Yves.Preaux@free.fr

We define the \mathbf{H}_s norms by

$$(2.2) \quad \|\mathbf{w}\|_s^2 = \sum_{\mathbf{k}} |\mathbf{k}|^{2s} |\widehat{\mathbf{w}}(\mathbf{k}, t)|^2,$$

where of course $\|\mathbf{w}\|_0^2 = \|\mathbf{w}\|^2$. It can be shown that when s is an integer, $\|\mathbf{w}\|_s^2 = \|\nabla^s \mathbf{w}\|^2$ (see [3]).

We denote by P_L The Helmholtz-Leray orthogonal projection of $(L^2)^3$ onto \mathbf{H}_0 and by A the Stokes operator defined by $A = -P_L \Delta$ on $D(A) = \mathbf{H}_0 \cap (H^2)^3$. We note that in the space-periodic case $A\mathbf{w} = -\Delta \mathbf{w}$ for all $\mathbf{w} \in D(A)$.

The operator A^{-1} is a self-adjoint positive definite compact operator from \mathbf{H}_s onto \mathbf{H}_s , for $s = 1$ and $s = 2$ (see [5]). We denote λ_1 the smallest eigenvalue of A .

We introduce the trilinear form b defined by

$$(2.3) \quad b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j} \int_{\Omega} u_i \partial_i v_j w_j dx.$$

wherever the integrals make sense. Note that $b(\mathbf{u}, \mathbf{w}, \mathbf{w}) = 0$ when $\nabla \cdot \mathbf{u} = 0$.

2.2 The Filter and the deconvolution process

Let $\mathbf{w} \in \mathbf{H}_0$ and $\overline{\mathbf{w}} \in \mathbf{H}_1$ be the unique solution to the following Stokes problem with periodic boundary conditions:

$$(2.4) \quad -\delta^2 \Delta \overline{\mathbf{w}} + \overline{\mathbf{w}} + \nabla r = \mathbf{w} \quad \text{in } \mathbb{R}^3, \quad \nabla \cdot \overline{\mathbf{w}} = 0, \quad \int_{\Omega} \overline{\mathbf{w}} = \mathbf{0}.$$

We denote the filtering operation by G so that $\overline{\mathbf{w}} = G\mathbf{w}$. Writing $\mathbf{w}(\mathbf{x}, t) = \sum_{\mathbf{k}} \widehat{\mathbf{w}}(\mathbf{k}, t) e^{-i\mathbf{k} \cdot \mathbf{x}}$, it is easily seen that $\nabla r = 0$ and $\overline{\mathbf{w}}(\mathbf{x}, t) = \sum_{\mathbf{k}} \frac{\widehat{\mathbf{w}}(\mathbf{k}, t)}{1 + \delta^2 |\mathbf{k}|^2} e^{-i\mathbf{k} \cdot \mathbf{x}}$.

Then writing $\overline{\mathbf{w}} = G(\mathbf{w})$, we see that in the corresponding spaces of the type \mathbf{H}_s , the transfer function of G , denoted by \widehat{G} , is the function $\widehat{G}(\mathbf{k}) = \frac{1}{1 + \delta^2 |\mathbf{k}|^2}$, and we also can write on the \mathbf{H}_s type spaces

$$(2.5) \quad -\delta^2 \Delta \overline{\mathbf{w}} + \overline{\mathbf{w}} = \mathbf{w} \quad \text{in } \mathbb{R}^3, \quad \nabla \cdot \overline{\mathbf{w}} = 0, \quad \int_{\Omega} \overline{\mathbf{w}} = \mathbf{0}.$$

The procedure of deconvolution by the Van Citter approximation is described in [6]. This yields the operator $D_N \mathbf{w} = \sum_{n=0}^N (I - G)^n \mathbf{w}$.

Definition 2.1 *The truncation operator $H_N : \mathbf{H}_s \rightarrow \mathbf{H}_s$ is defined by $H_N \mathbf{w} := D_N \overline{\mathbf{w}} = (D_N \circ G) \mathbf{w}$.* ■

Note that, for any $s \geq 0$ we have the following proprieties (see [6]) :

$$(2.6) \quad \|H_N \mathbf{w}\|_s \leq \|\mathbf{w}\|_s, \quad \|H_N \mathbf{w}\|_{s+2} \leq C(\delta, N) \|\mathbf{w}\|_s.$$

2.3 The model

Let $\mathbf{u}_0 \in \mathbf{H}_0$, $f \in \mathbf{H}_{-1}$. For $\delta > 0$, let the averaging be defined by (2.4). The problem we consider is the following: for a fixed $T > 0$, find (\mathbf{w}, q)

$$(2.7) \quad \begin{cases} \mathbf{w} \in L^2([0, T], \mathbf{H}_1) \cap L^\infty([0, T], \mathbf{H}_0), & \partial_t \mathbf{w} \in L^2([0, T], \mathbf{H}_{-1}) \\ q \in L^2([0, T], L^2_{\text{per},0}), \\ \partial_t \mathbf{w} + (H_N(\mathbf{w}) \cdot \nabla) \mathbf{w} - \nu \Delta \mathbf{w} + \nabla q = H_N(\mathbf{f}) & \text{in } \mathcal{D}'([0, T] \times \mathbb{R}^3), \\ \mathbf{w}(\mathbf{x}, 0) = H_N(\mathbf{u}_0) = \mathbf{w}_0. \end{cases}$$

where $L^2_{\text{per},0}$ denotes the scalar fields in $L^2_{\text{loc}}(\mathbb{R}^3)$, 2π -periodic with zero mean value. We prove in [6] the following result.

Theorem 2.1 *Problem (2.7) admits a unique solution (\mathbf{w}, q) , $\mathbf{w} \in L^\infty([0, T], \mathbf{H}_1) \cap L^2([0, T], \mathbf{H}_2)$, and the following energy equality holds:*

$$(2.8) \quad \frac{1}{2} \|\mathbf{w}(t)\|^2 + \nu \int_0^t \int_\Omega |\nabla \mathbf{w}|^2 d\mathbf{x} dt' = \frac{1}{2} \|H_N(\mathbf{u}_0)\|^2 + \int_0^t \int_\Omega H_N(\mathbf{f}) \cdot \mathbf{w} d\mathbf{x} dt'. \quad \blacksquare$$

3 Main result

3.1 Recall of basic notions about attractors

We denote by $\mathbf{w}(t, \cdot) = S(t)(\mathbf{w}_0)$ the (unique) solution of system (2.7) at time t . We recall the definitions of a global attractor and an absorbing set (see in [8]).

Definition 3.1 *We say that $\mathcal{A} \subset \mathbf{H}_0$ is a global attractor for the dynamical system (2.7) if and only if*

(P1) \mathcal{A} is compact in the space \mathbf{H}_0 ,

(P2) $\forall t \in \mathbb{R}, S(t)(\mathcal{A}) \subset \mathcal{A}$,

(P3) *For every bounded subset $B \subset \mathbf{H}_0$, $\rho(S(t)(B), \mathcal{A})$ goes to zero when t goes to infinity, where $\rho(S(t)(B), \mathcal{A}) = \sup_{v \in B} \inf_{u \in \mathcal{A}} \|u - v\|$.* \blacksquare

Definition 3.2 1. *A set $A \subset \mathbf{H}_0$ is an absorbing set if and only if for every bounded subset $B \subset \mathbf{H}_0$ there exists $t_1 > 0$ such that for all $t \geq t_1$ one has $S(t)(B) \subset A$.*

2. *We say that the semi group $S(t)$ is uniformly compact if and only if for every bounded subset $B \subset \mathbf{H}_0$ there exists $t_2 = t_2(B)$ such that $\overline{\bigcup_{t \geq t_2} S(t)(B)}$ is compact.*

3. *We denote by $\omega(A)$ the set $\omega(A) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)(A)}$.* \blacksquare

Proposition 3.1 *Assume that there exists an absorbing bounded set A and that the semi group $S(t)$ is uniformly compact, then $\mathcal{A} = \omega(A)$ is the global attractor for the dynamical system defined by $S(t)$.*

see the proof in [8]).

3.2 Existence of a global attractor

We are now in order to state and prove the main result of this note.

Theorem 3.1 *The system (2.7) has a global attractor.* ■

Proof. Thanks to Proposition 3.1, it remains to prove that system (2.7) has an absorbing set and that $S(t)$ is uniformly compact, in the sense of definition 3.2. Both things are derived from basic estimates that we detail in the following.

Absorbing set in \mathbf{H}_0 : We take the inner product of the first quation of system (2.7) with \mathbf{w} to obtain

$$(3.1) \quad \frac{1}{2} \frac{d}{dt} \|\mathbf{w}\|^2 + b(H_N(\mathbf{w}), \mathbf{w}, \mathbf{w}) + \nu \|\mathbf{w}\|_1^2 = (H_N(\mathbf{f}), \mathbf{w}).$$

Observing that $b(H_N(\mathbf{w}), \mathbf{w}, \mathbf{w}) = 0$ due to $\nabla \cdot H_N(\mathbf{w}) = 0$, applying Young inequality, Poincare inequality $\|\mathbf{w}\| \leq \lambda_1^{-\frac{1}{2}} \|\mathbf{w}\|_1$ and using (2.6) there remains

$$(3.2) \quad \frac{d}{dt} \|\mathbf{w}\|^2 + \nu \lambda_1 \|\mathbf{w}\|^2 \leq \frac{1}{\nu \lambda_1} \|\mathbf{f}\|^2.$$

So, noting $\rho_0 = \frac{1}{\nu \lambda_1} \|\mathbf{f}\|^2$ and applying Gronwall lema we obtain

$$(3.3) \quad \|\mathbf{w}\|^2 \leq \|\mathbf{w}_0\|^2 e^{-\nu \lambda_1 t} + \rho_0^2 (1 - e^{-\nu \lambda_1 t}).$$

Considering \mathbf{w}_0 included in a ball $B(0, R)$ and choosing $\rho'_0 > \rho_0$, the previous inequality implies that, for $t > T_0$,

$$(3.4) \quad \|\mathbf{w}(t)\|^2 < \rho_0'^2, \quad \text{with} \quad T_0 = \frac{1}{\nu \lambda_1} \ln \frac{R^2}{\rho_0'^2 - \rho_0^2}.$$

Since each bounded set of \mathbf{H}_0 is included in a ball $B(0, R)$, one deduces that $B(0, \rho'_0)$ is an absorbing set in \mathbf{H}_0 .

More, as an alternative of (3.2) we may obtain

$$(3.5) \quad \frac{d}{dt} \|\mathbf{w}\|^2 + \nu \|\mathbf{w}\|_1^2 \leq \frac{1}{\nu \lambda_1} \|\mathbf{f}\|^2.$$

Integrating between t and $t + r$, we observe than, for $\mathbf{u}_0 \in B(0, R)$, $\rho'_0 > \rho_0$ and $t > T_0$ (with $T_0 = \frac{1}{\nu \lambda_1} \ln \frac{R^2}{\rho_0'^2 - \rho_0^2}$) :

$$(3.6) \quad \int_t^{t+r} \|\mathbf{w}(s)\|_1^2 ds \leq \frac{r}{\nu^2 \lambda_1} \|\mathbf{f}\|^2 + \frac{\rho_0'^2}{\nu}.$$

Absorbing set in \mathbf{H}_1 : We take now the inner product of the first equation of system (2.7) with $A\mathbf{w}$ to obtain

$$(3.7) \quad \frac{1}{2} \frac{d}{dt} \|\mathbf{w}\|_1^2 + b(H_N(\mathbf{w}), \mathbf{w}, A\mathbf{w}) + \nu \|A\mathbf{w}\|^2 = (H_N(\mathbf{f}), A\mathbf{w}),$$

leading to

$$(3.8) \quad \frac{1}{2} \frac{d}{dt} \|\mathbf{w}\|_1^2 + \nu \|A\mathbf{w}\|^2 \leq \frac{1}{\nu} \|H_N(\mathbf{f})\|^2 + \frac{\nu}{4} \|A\mathbf{w}\|^2 + |b(H_N(\mathbf{w}), \mathbf{w}, A\mathbf{w})|,$$

The trilinear form b satisfies the following inequality (see in [6]) :

$$(3.9) \quad |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq c' \|\mathbf{u}\|^{1/4} \|\mathbf{u}\|_1^{3/4} \|\mathbf{v}\|_1^{1/4} \|A\mathbf{v}\|^{3/4} \|\mathbf{w}\|.$$

Therefore, one has

$$(3.10) \quad |b(H_N(\mathbf{w}), \mathbf{w}, A\mathbf{w})| \leq c' \|H_N(\mathbf{w})\|^{1/4} \|H_N(\mathbf{w})\|_1^{3/4} \|\mathbf{w}\|_1^{1/4} \|A\mathbf{w}\|^{7/4}.$$

Using (2.6) we have $\|H_N(\mathbf{w})\|_1 \leq \|H_N(\mathbf{w})\|_2 \leq C(\delta, N) \|\mathbf{w}\|$ and using (2.6) :

$$(3.11) \quad |b(H_N(\mathbf{w}), \mathbf{w}, A\mathbf{w})| \leq C'(\delta, N) \|\mathbf{w}\| \|\mathbf{w}\|_1^{1/4} \|A\mathbf{w}\|^{7/4}.$$

By Young inequality we obtain

$$(3.12) \quad |b(H_N(\mathbf{w}), \mathbf{w}, A\mathbf{w})| \leq \frac{\nu}{4} \|A\mathbf{w}\|^2 + \frac{C_1(\delta, N)}{2} \|\mathbf{w}\|^8 \|\mathbf{w}\|_1^2,$$

thus

$$(3.13) \quad \frac{d}{dt} \|\mathbf{w}\|_1^2 + \nu \|A\mathbf{w}\|^2 \leq \frac{2}{\nu} \|H_N(\mathbf{f})\|^2 + C_1(\delta, N) \|\mathbf{w}\|^8 \|\mathbf{w}\|_1^2$$

We now use a Gronwall type proposition (see the proof in [8]):

Proposition 3.2 *Assume that y , g and h are positive, locally integrable functions on $]t_0, +\infty[$, and that for $t \geq t_0$,*

$$\frac{dy}{dt} \leq gy + h, \quad \int_t^{t+r} y(s)ds \leq k_1, \quad \int_t^{t+r} g(s)ds \leq k_2, \quad \int_t^{t+r} h(s)ds \leq k_3,$$

where r, k_1, k_2, k_3 are four positive constants, then

$$y(t+r) \leq \left(\frac{k_1}{r} + k_3 \right) e^{k_2}, \quad \forall t \geq t_0. \quad \blacksquare$$

We can now finish the proof. Thanks to (3.4) and (3.6), using this lemma with $y = \|\mathbf{w}\|_1^2$, $g = C_1(\delta, N) \|\mathbf{w}\|^8$ and $h = \frac{2}{\nu} \|H_N(\mathbf{f})\|^2$, we obtain,

$$(3.14) \quad \|\mathbf{w}(t)\|_1^2 \leq \left(\frac{k_1}{r} + k_3 \right) e^{k_2}, \quad \forall t \geq T_0 + r,$$

with $k_1 = \frac{r}{\nu^2 \lambda_1} \|\mathbf{f}\|^2 + \frac{1}{\nu} \rho_0'^2$, $k_2 = C_1(\delta, N) \rho_0'^8$, $k_3 = \frac{2r}{\nu} \|\mathbf{f}\|^2$.

Thus, after a time $T_1 = T_1(\|\mathbf{w}_0\|, \|\mathbf{f}\|, \nu)$, \mathbf{w} is included in a ball of radius $R = R(\|\mathbf{f}\|, \nu, \delta, N)$. One deduces that there exists an absorbing set in \mathbf{H}_1 .

Let B be a bounded set in \mathbf{H}_1 . Estimate (3.14) implies that $\bigcup_{t \geq T_0+r} S(t)B$ is a bounded

set in \mathbf{H}_1 which is compactly imbedded in \mathbf{H}_0 , so $S(t)$ is uniformly compact. Estimate (3.14) also implies the existence of an absorbing bounded set since k_1, k_2 and k_3 are independent of \mathbf{w}_0 . Thanks to (3.1), this achieves the proof of the theorem. \blacksquare

References

- [1] V. V. CHEPYZHOV, E. S. TITI, AND M. I. VISHIK, *On the convergence of the leray-alpha model to the trajectory attractor of the 3D Navier-Stokes system*, Matematicheskii Sbornik, 12 (2007), pp. 3–36.
- [2] A. CHESKIDOV, D. D. HOLM, E. OLSON, AND E. S. TITI, *On a leray- α model of turbulence*, Royal Society London, Proceedings, Series A, Mathematical, Physical and Engineering Sciences, 461 (2005), pp. 629–649.
- [3] C. DOERING AND J. GIBBON, *Applied analysis of the Navier-Stokes equations*, Cambridge University Press, 1995.
- [4] C. FOIAS, D. D. HOLM, AND E. S. TITI, *The Navier-Stokes-alpha model of fluid turbulence*, Physica D, 152 (2001), pp. 505–519.
- [5] C. FOIAS, O. MANLEY, R. ROSA, AND R. TEMAM, *Navier-Stokes Equations and Turbulence*, Cambridge University Press, 2001.
- [6] W. LAYTON AND R. LEWANDOWSKI, *A high accuracy leray-deconvolution model of turbulence and its limiting behavior*, Analysis and Applications, 6 (2008), pp. 1–27.
- [7] S. STOLZ, N. A. ADAMS, AND L. KLEISER, *An approximate deconvolution model for large-eddy simulation with application to incompressible wall-bounded flows*, Physics of fluids, 13 (2001), pp. 997–1015.
- [8] R. TEMAM, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer Verlag, 1988.